

Assignment 9
Solutions

Problem 1

We are asking what is the maximum $Q^2$ value for this scattering process. For a beam with energy $E$ hitting a target at rest, $Q^2$ is given by Eq. 6.11 (p. 203): 

$$Q^2 = 4EE' \sin^2 \frac{\theta}{2},$$

where $\theta$ is the scattering angle and $E'$ the scattered particle energy. It is obvious that the maximum 4-momentum transfer squared is for backward scattering ($\theta = 180^\circ$); clearly, forward scattering ($\theta = 0^\circ$) is equivalent to no scattering at all, so no energy-momentum transfer. Therefore, $Q^2_{\text{max}} = 4EE'$.

We will also assume elastic scattering. We can then use $Q^2 = 2M\nu = 2M(E - E')$, valid for elastic scattering, to get rid of $E'$. (This is the same formula as for deep-inelastic scattering, but with the Bjorken-$x$ value set to $x = 1$; see also Eq. 1.60 on p. 20.) We get

$$E' = E - \frac{Q^2}{2M}.$$ 

Substituting, we find

$$Q^2_{\text{max}} = \frac{4E^2M}{M + 2E} = \frac{4(2 \text{ GeV})^2(52 \text{ GeV})}{52 \text{ GeV} + 2(2 \text{ GeV})} = 14.9 \text{ GeV}^2.$$

Here we used the $^{56}\text{Fe}$ atomic weight, 55.9 amu, and the conversion 1 amu = 931.5 MeV to find $M = 52.0 \text{ GeV}$.

If it is not obvious that $\theta = 180^\circ$ results in the highest possible $Q^2$, we can repeat the calculation without this assumption and verify that the final formula for $Q^2$ has a maximum value for this angle.

Problem 2

The Rutherford formula, Eq. 1.70 (p. 22), integrated over the polar angle $\phi$ gives (Eq. 1.71)

$$\frac{d\sigma}{d \cos \theta} = \frac{A}{(1 - \cos \theta)^2},$$ 

where the factor $A$ is independent of $\theta$. Since we are only interested in the ratio of two ranges in $\theta$, we do not care about the exact form of $A$. We get

$$\frac{\sigma(\theta > 90^\circ)}{\sigma(\theta > 10^\circ)} = \frac{\int_{90^\circ}^{180^\circ} d\sigma}{\int_{10^\circ}^{180^\circ} d\sigma} = \frac{\int_{90^\circ}^{180^\circ} \frac{d\cos \theta}{(1-\cos \theta)^2}}{\int_{10^\circ}^{180^\circ} \frac{d\cos \theta}{(1-\cos \theta)^2}} = \frac{1}{\int_{90^\circ}^{180^\circ} \frac{1}{1-\cos \theta}} \frac{1}{\int_{10^\circ}^{180^\circ} \frac{1}{1-\cos \theta}} = 0.0074.$$
Note: the integral is easy to do by writing $d \cos \theta = -d(1 - \cos \theta)$ and then substituting $x \equiv 1 - \cos \theta$.

**Problem 3**

We will show that, at any given scattering angle, the maximum scattered-electron energy is obtained for elastic scattering. Otherwise this is very similar to the first problem, except that we will eliminate $Q^2$ and solve for $E'$.

For inelastic scattering, the Bjorken-$x$ variable is defined as $x = \frac{Q^2}{2M_\nu}$ and, as we saw, it represents the momentum fraction of the nucleon carried by the struck quark in the infinite momentum frame, so $x$ is in the range 0 to 1; $x = 1$ corresponds to elastic scattering (or deep-inelastic scattering off a quark carrying the entire momentum of the nucleon). Therefore we have $Q^2 = 2Mx\nu = 2Mx(E - E')$. We also have $Q^2 = 4EE'\sin^2 \frac{\theta}{2}$. From these two we get

$$2Mx(E - E') = 4EE'\sin^2 \frac{\theta}{2},$$

which we can solve for $E'$ to get the more general form of Eq. 1.60:

$$E' = \frac{E}{1 + \frac{2E}{Mx} \sin^2 \frac{\theta}{2}}.$$

It is clear that $x = 1$ (elastic scattering) minimizes the second term in the denominator and therefore maximizes $E'$. Substituting the given values we get

$$E' = \frac{10 \text{ GeV}}{1 + \frac{2(10 \text{ GeV})}{0.938 \text{ GeV}} \sin^2 (15^\circ)} = 4.12 \text{ GeV}.$$

**Problem 4**

This is a straightforward substitution into the formulas we encountered in the previous problem. From $Q^2 = 2Mx(E - E')$ we get

$$E' = E - \frac{Q^2}{2Mx} = 100 \text{ GeV} - \frac{25 \text{ GeV}^2}{2(0.938 \text{ GeV})0.2} = 33.4 \text{ GeV}.$$

**Problem 5**

The center-of-mass energy is given by $\sqrt{s} = \sqrt{(E_e + E_p)^2 - (p_e + p_p)^2}$. All the energies concerned are much larger than the corresponding particle masses, so we will set $p \approx E$. Taking into account that the electron and proton energies point in opposite directions, we have

$$\sqrt{s} = \sqrt{(820 + 30)^2 - (820 - 30)^2} \text{ GeV} = 313.7 \text{ GeV}.$$
In a fixed-target experiment with an electron beam hitting a proton target, the above expression becomes \( \sqrt{s} = \sqrt{2m_p E_e} \), as we have seen. In order to reach the same center-of-mass energy, the electron-beam energy would have to be

\[
E_e = \frac{s}{2m_p} = \frac{98400 \text{ GeV}^2}{2(0.938 \text{ GeV})} = 52.5 \text{ TeV},
\]

far beyond the capabilities of any current or near-future machine.

To calculate the maximum \( Q^2 \) for the given \( x \) values, we again start from \( Q^2 = 2m \nu x \) (this is the definition of the Bjorken-\( x \) variable) but we should be careful with \( \nu \): the expression \( \nu = E - E' \) that we used before is only valid for fixed-target experiments. The general definition of \( \nu \) is

\[
\nu = \frac{P \cdot q}{M},
\]

where \( P \) and \( q \) are the target and virtual-photon 4-vectors, respectively. (Here the proton is considered to be the target, even though it is in motion, since it is the object being studied with the electron beam.) We can easily show that the expression \( \nu = E_e - E'_e \) is obtained when the target is at rest. In the HERA situation, we have \( P = (E_p, p_p) \) and \( q = (E_e - E'_e, p_e - p'_e) \). We can verify that \( Q^2 = 4EE'\sin^2 \frac{\theta}{2} \) is still valid, as long as the scattering angle \( \theta \) is still defined relative to the electron’s initial momentum direction. (Recall that \( Q^2 \equiv -q^2 \).) We could also find \( \nu \) in the lab frame, but there is a simpler way.

We note that \( q^2 \), and therefore also \( Q^2 \), is a relativistic invariant, as the square of a 4-vector, so we can choose to calculate its value in any frame where it is most convenient. We already have done similar calculations in the rest frame of the proton, so we could pick this frame. We calculated above that this experiment is equivalent to a fixed-target experiment as long as the electron-beam energy is \( E_e = 52.5 \text{ TeV} \). Therefore this is the electron momentum in the proton rest frame. Continuing the calculation in this frame, where \( \nu = E_e - E'_e \), we get

\[
Q^2 = 4E_e(E_e - \nu)\sin^2 \frac{\theta}{2}.
\]

Combining with \( Q^2 = 2m \nu x \) to eliminate \( \nu \), we get

\[
Q^2 = \frac{4E_e^2 \sin^2 \frac{\theta}{2}}{1 + \frac{2E_e}{m_p x} \sin^2 \frac{\theta}{2}}.
\]

It is clear that the maximum value of \( Q^2 \) for a given \( x \) is again obtained for \( \theta = 180^\circ \) (the electron is back-scattered; verify by dividing numerator and denominator by \( \sin^2 \frac{\theta}{2} \)), while for \( \theta = 0 \) we get \( Q^2 = 0 \). Then

\[
Q^2_{\text{max}} = \frac{4E_e^2}{1 + \frac{2E_e}{m_p x}} \approx 2E_e m_p x.
\]

Using \( E = 52.5 \times 10^3 \text{ GeV} \) and \( m_p = 0.938 \text{ GeV} \) we get for the various values of \( x \):

\[
\begin{array}{cc}
x & Q^2_{\text{max}} \\
0.4 & 3.94 \times 10^4 \text{ GeV}^2 \\
0.01 & 984 \text{ GeV}^2 \\
0.0001 & 9.84 \text{ GeV}^2
\end{array}
\]
Looking at Fig. 6.13 (p. 209), we see that the data for these $x$ values approach, but not quite reach, the maximum $Q^2$ values. This is because the number of events decreases as one approaches the kinematic limits and also the cross section decreases sharply as $Q^2$ increases.

**Problem 6**

Using Eq. 6.55 (p. 219) for $\alpha_s$ and Eq. 5.38 (p. 190) for $\alpha$, we find for this ratio

$$\frac{\alpha(Q^2)}{\alpha_s(Q^2)} = \frac{1}{12\pi} (33 - 2n_f) \ln(|Q^2|/\Lambda^2_{\text{QCD}}) = \frac{0.61 \ln(|Q^2|/0.2^2)}{128 - 0.71 \ln(|Q^2|/91^2)},$$

where $Q$ is in GeV, for $n_f = 5$ (five active quark flavors in this range) and $z_f = 6.67$ (see p. 189 for explanation). For $|Q| = 10$ GeV this becomes

$$\frac{\alpha}{\alpha_s} = 0.036,$$

while for $|Q| = 100$ GeV we get

$$\frac{\alpha}{\alpha_s} = 0.059.$$

**Problem 7**

1. We use the definitions of $F_2(x)$ in terms of parton distributions for charged-lepton and neutrino scattering, Eqs. 6.23 and 6.27, respectively (p. 207). Averaging over proton and neutron targets we get for the nucleon structure functions

$$\frac{F_2^{\ell N}(x)}{x} = \frac{5}{18} \left[ u(x) + \bar{u}(x) + d(x) + \bar{d}(x) \right] + \frac{1}{9} \left[ s(x) + \bar{s}(x) \right]$$

and

$$\frac{F_2^{\nu N}(x)}{x} = \left[ u(x) + \bar{u}(x) + d(x) + \bar{d}(x) + s(x) + \bar{s}(x) \right].$$

Note that the book omits the strange quarks from the neutrino structure functions. We added them in here: the neutrino interacts with the strange antiquark and the antineutrino with a strange quark, in order to conserve charge when they turn into the corresponding charged lepton or antilepton.

(a) Ignoring the strange quarks, we immediately see that

$$F_2^{\ell N}(x) = \frac{5}{18} F_2^{\nu N}(x).$$

(b) If we include strange quarks, then we have

$$F_2^{\nu N}(x) - \frac{18}{5} F_2^{\ell N}(x) = \frac{3}{5} x [s(x) + \bar{s}(x)].$$

We see that the difference between neutrino and charged-lepton structure functions, with the appropriate $\frac{18}{5}$ factor for the latter, gives a direct measurement of the strange quark-antiquark sea in the nucleon.
2. Again using the definitions of $F_2^{ep}(x)$ and $F_2^{en}(x)$ in Eqs. 6.23 we find

$$\frac{1}{x} \left[ F_2^{ep}(x) - F_2^{en}(x) \right] = \frac{1}{3} [u(x) - d(x)] + \frac{1}{3} \left[ \bar{u}(x) - \bar{d}(x) \right],$$

assuming that the strange quark content is the same in the proton and the neutron. Using the definition of valence quarks $q_V(x) \equiv q(x) - \bar{q}(x)$, we can rewrite this as

$$\frac{1}{x} \left[ F_2^{ep}(x) - F_2^{en}(x) \right] = \frac{1}{3} [u_V(x) - d_V(x)] + \frac{2}{3} \left[ \bar{u}(x) - \bar{d}(x) \right],$$

adding and subtracting the sea up and down antiquarks times one-third. The reason is that we know what the integrals over the valence quarks are over the entire $x$ range. We will use this in the following.

(a) Assuming that the sea is isospin symmetric, the last term is zero and we get

$$\int_0^1 \frac{dx}{x} \left[ F_2^{ep}(x) - F_2^{en}(x) \right] = \int_0^1 \frac{dx}{3} [u_V(x) - d_V(x)].$$

Since we have $\int_0^1 u_V(x)dx = 2$ and $\int_0^1 d_V(x)dx = 1$ (there are a total of two up and one down valence quarks in the nucleon), the Gottfried sum rule is

$$\int_0^1 \frac{dx}{x} \left[ F_2^{ep}(x) - F_2^{en}(x) \right] = \frac{1}{3}.$$

(b) If we don’t assume an isospin-symmetric sea, then we can write

$$\int_0^1 \frac{dx}{x} \left[ F_2^{ep}(x) - F_2^{en}(x) \right] = \frac{1}{3} - \frac{2}{3} \int_0^1 dx \left[ \bar{u}(x) - \bar{d}(x) \right].$$

We can see that any deviation of the integral on the left-hand side from the value of one-third implies that the sea is not isospin-symmetric. Experimentally, the sum rule is found to be violated, with a value smaller than $\frac{1}{3}$, implying that there are more down quark-antiquark pairs in the proton sea (and, accordingly, more up quark-antiquark pairs in the neutron). There are several possible explanations for this asymmetry, but the effect is still not fully understood. Note that this does not imply that isospin is not a good symmetry for the strong interaction or the structure of the nucleon; in fact, we implicitly assumed this symmetry above, that proton and neutron are mirror images of each other, so the structure of the one can be obtained with the substitution of up quarks with down and vice-versus. The excess of down pairs in the proton sea is probably connected to the presence of more up valence quarks combined with the Pauli exclusion principle; and similarly for the neutron.

3. The interpretation of $F_3$ in the quark-parton model is

$$F_3(x) = [u(x) - \bar{u}(x)] + [d(x) - \bar{d}(x)] = u_V(x) + d_V(x).$$

Then the integral simply counts the total number of valence quarks:

$$\int_0^1 dx F_3(x) = \int_0^1 u_V(x)dx + \int_0^1 d_V(x)dx = 2 + 1 = 3.$$